DAY ONE

Sets, Relations and Functions

Learning & Revision for the Day

- Sets
- Venn Diagram
- Operations on Sets
- Law of Algebra of Sets
- Cartesian Product of Sets
- Relations
- Composition of Relations
- Functions or Mapping
- Composition of Functions

Sets

- A **set** is a well-defined class or collection of the objects.
- Sets are usually denoted by the symbol $A, B, C, \ldots$ and its elements are denoted by a, b, c, … etc.
- If a is an element of a set A, then we write $a \in A$ and if not then we write $a \notin A$.

Representations of Sets

There are two methods of representing a set :

- In **roster method**, a set is described by listing elements, separated by commas, within curly braces$\{\neq\}$. e.g. A set of vowels of English alphabet may be described as $\{a, e, i, o, u\}$.
- In **set-builder method**, a set is described by a property $P(x)$, which is possessed by all its elements x. In such a case the set is written as $\{x : P(x) \text{ holds}\}$ or $\{x \mid P(x) \text{ holds}\}$, which is read as the set of all x such that $P(x)$ holds. e.g. The set $P = \{0, 1, 4, 9, 16, \ldots\}$ can be written as $P = \{x^2 \mid x \in Z\}$.

Types of Sets

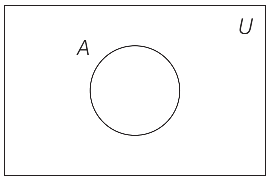
- The set which contains no element at all is called the **null set** (empty set or void set) and it is denoted by the symbol 'ϕ' or '$\{\}$' and if it contains a single element, then it is called **singleton set**.
- A set in which the process of counting of elements definitely comes to an end, is called a **finite set,** otherwise it is an **infinite set.**
- Two sets A and B are said to be **equal set** iff every element of A is an element of B and also every element of B is an element of A. i.e. $A = B$, if $x \in A \Leftrightarrow x \in B$.

- **Equivalent sets** have the same number of elements but not exactly the same elements.
- A set that contains all sets in a given context is called **universal set** (U).
- Let A and B be two sets. If every element of A is an element of B, then A is called a **subset** of B, i.e. $A \subseteq B$.
- If A is a subset of B and $A \neq B$, then A is a **proper subset** of B. i.e. $A \subset B$.
- The null set ϕ is a subset of every set and every set is a subset of itself i.e. $\phi \subset A$ and $A \subseteq A$ for every set A. They are called **improper subsets** of A.
- If S is any set, then the set of all the subsets of S is called the **power set** of S and it is denoted by $P(S)$. Power set of a given set is always non-empty. If A has n elements, then $P(A)$ has 2^n elements.

NOTE
- The set $\{\phi\}$ is not a null set. It is a set containing one element ϕ.
- Whenever we have to show that two sets A and B are equal show that $A \subseteq B$ and $B \subseteq A$.
- If a set A has m elements, then the number m is called **cardinal number** of set A and it is denoted by $n(A)$. Thus, $n(A) = m$.

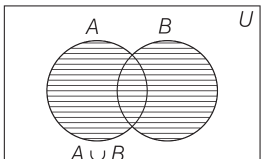
Venn Diagram

The combination of rectangles and circles is called **Venn Euler diagram** or Venn diagram. In Venn diagram, the universal set is represented by a rectangular region and a set is represented by circle on some closed geometrical figure. Where, A is the set and U is the universal set.



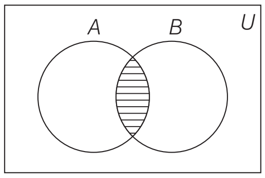
Operations on Sets

- The **union** of sets A and B is the set of all elements which are in set A or in B or in both A and B.
  i.e. $A \cup B = \{x : x \in A \text{ or } x \in B\}$



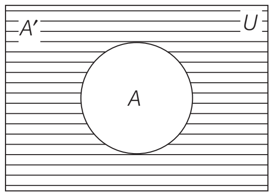
$A \cup B$

- The **intersection** of A and B is the set of all those elements that belong to both A and B.
  i.e. $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

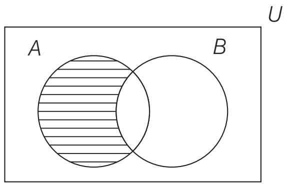


$A \cap B$

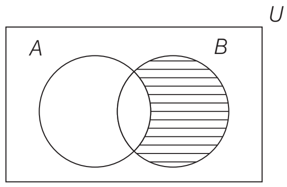
- If $A \cap B = \phi$, then A and B are called **disjoint sets**.
- Let U be an universal set and A be a set such that $A \subset U$. Then, **complement of A** with respect to U is denoted by A' or A^c or $\overline{A}$ or $U - A$. It is defined as the set of all those elements of U which are not in A.



- The **difference** $A - B$ is the set of all those elements of A which does not belong to B.
  i.e. $A - B = \{x : x \in A \text{ and } x \notin B\}$
  and $B - A = \{x : x \in B \text{ and } x \notin A\}$.

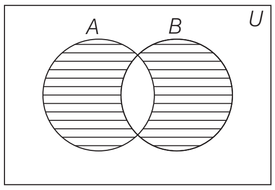


$A - B$



$B - A$

- The symmetric difference of sets A and B is the set $(A - B) \cup (B - A)$ and is denoted by $A \Delta B$.
  i.e. $A \Delta B = (A - B) \cup (B - A)$



$A \Delta B$

Law of Algebra of Sets

If A, B and C are any three sets, then

1. **Idempotent Laws**
   (i) $A \cup A = A$ (ii) $A \cap A = A$
2. **Identity Laws**
   (i) $A \cup \phi = A$ (ii) $A \cap U = A$
3. **Distributive Laws**
   (i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

4. **De-Morgan's Laws**

(i) $(A \cup B)' = A' \cap B'$

(ii) $(A \cap B)' = A' \cup B'$

(iii) $A - (B \cap C) = (A - B) \cup (A - C)$

(iv) $A - (B \cup C) = (A - B) \cap (A - C)$

5. **Associative Laws**

(i) $(A \cup B) \cup C = A \cup (B \cup C)$

(ii) $A \cap (B \cap C) = (A \cap B) \cap C$

6. **Commutative Laws**

(i) $A \cup B = B \cup A$ (ii) $A \cap B = B \cap A$

(iii) $A \Delta B = B \Delta A$

Important Results on Operation of Sets

1. $A - B = A \cap B'$
2. $B - A = B \cap A'$
3. $A - B = A \Leftrightarrow A \cap B = \phi$
4. $(A - B) \cup B = A \cup B$
5. $(A - B) \cap B = \phi$
6. $A \subseteq B \Leftrightarrow B' \subseteq A'$
7. $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$
8. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
9. $n(A \cup B) = n(A) + n(B)$
   $\Leftrightarrow A$ and B are disjoint sets.
10. $n(A - B) = n(A) - n(A \cap B)$
11. $n(A \Delta B) = n(A) + n(B) - 2n(A \cap B)$
12. $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B)$
    $- n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$
13. $n(A' \cup B') = n(A \cap B)' = n(U) - n(A \cap B)$
14. $n(A' \cap B') = n(A \cup B)' = n(U) - n(A \cup B)$

Cartesian Product of Sets

Let A and B be any two non-empty sets. Then the cartesian product $A \times B$, is defined as set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$.
i.e.

- $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$
- $B \times A = \{(b, a) : b \in B \text{ and } a \in A\}$
  and $A \times A = \{(a, b) : a, b \in A\}$.
- $A \times B = \phi$, if either A or B is an empty set.
- If $n(A) = p$ and $n(B) = q$, then
  $n(A \times B) = n(A) \cdot n(B) = pq$.
- $A \times (B \cup C) = (A \times B) \cup (A \times C)$
- $A \times (B \cap C) = (A \times B) \cap (A \times C)$
- $A \times (B - C) = (A \times B) - (A \times C)$
- $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

Relations

- Let A and B be two non-empty sets, then **relation** R from A to B is a subset of $A \times B$, i.e. $R \subseteq A \times B$.
- If $(a, b) \in R$, then we say a is related to b by the relation R and we write it as aRb.
- Domain of $R = \{a : (a, b) \in R\}$ and range of $R = \{b : (a, b) \in R\}$.
- If $n(A) = p$ and $n(B) = q$, then the total number of relations from A to B is 2^{pq}.

Types of Relations

- Let A be any non-empty set and R be a relation on A. Then,

(i) R is said to be **reflexive** iff $(a, a) \in R, \forall a \in A$.

(ii) R is said to be **symmetric** iff

$$(a, b) \in R$$
$$\Rightarrow \quad (b, a) \in R, \forall a, b \in A$$

(iii) R is said to be a **transitive** iff $(a, b) \in R$ and $(b, c) \in R$

$$\Rightarrow \quad (a, c) \in R, \forall a, b, c \in A$$

i.e. aRb and $bRc \Rightarrow aRc, \forall a, b, c \in A$.

- The relation $I_A = \{(a, a) : a \in A\}$ on A is called the **identity relation** on A.
- R is said to be an **equivalence relation** iff

(i) it is reflexive i.e. $(a, a) \in R, \forall a \in A$.

(ii) it is symmetric i.e. $(a, b) \in R \Rightarrow (b, a) \in R, \forall a, b \in A$

(iii) it is transitive

i.e. $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow (a, c) \in R, \forall a, b, c \in A$

Inverse Relation

Let R be a relation from set A to set B, then the **inverse of R**, denoted by R^{-1}, is defined by

$R^{-1} = \{(b, a) : (a, b) \in R\}$. Clearly, $(a, b) \in R \Leftrightarrow (b, a) \in R^{-1}$.

NOTE
- The intersection of two equivalence relations on a set is an equivalence relation on the set.
- The union of two equivalence relations on a set is not necessarily an equivalence relation on the set.
- If R is an equivalence relation on a set A, then R^{-1} is also an equivalence relation A.

Composition of Relations

Let R and S be two relations from set A to B and B to C respectively, then we can define a relation SoR from A to C such that $(a, c) \in SoR \Leftrightarrow \exists\, b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. This relation is called the **composition of R and S**.

$$RoS \neq SoR$$

Functions or Mapping

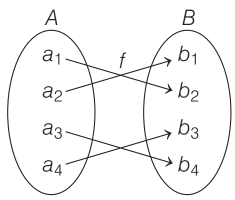
- If A and B are two non-empty sets, then a rule f which associates each $x \in A$, to a unique member $y \in B$, is called a function from A to B and it is denoted by $f : A \rightarrow B$.
- The set A is called the **domain** of $f(D_f)$ and set B is called the **codomain** of $f(C_f)$.
- The set consisting of all the f-images of the elements of the domain A, called the range of $f(R_f)$.

NOTE
- A relation will be a function, if no two distinct ordered pairs have the same first element.
- Every function is a relation but every relation is not necessarily a function.
- The number of functions from a finite set A into finite set B is $\{n(B)\}^{n(A)}$.

Different Types of Functions

Let f be a function from A to B, i.e. $f : A \rightarrow B$. Then,

f is said to be **one-one function** or injective function, if different elements of A have different images in B.

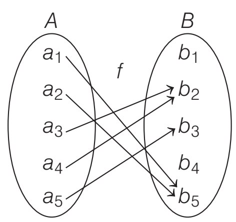


Methods to Check One-One Function

Method I If $f(x) = f(y) \Rightarrow x = y$, then f is one-one.

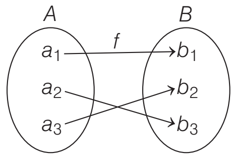
Method II A function is one-one iff no line parallel to X-axis meets the graph of function at more than one point.

- The number of one-one function that can be defined from a finite set A into finite set B is $\begin{cases} {}^{n(B)}P_{n(A)}, & \text{if } n(B) \geq n(A) \\ 0, & \text{otherwise} \end{cases}$.
- f is said to be a **many-one function**, if two or more elements of set A have the same image in B.



i.e. $f : A \rightarrow B$ is a many-one function, if it is not a one-one function.

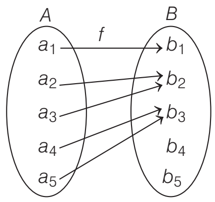
- f is said to be **onto function** or **surjective function**, if each element of B has its pre-image in A.



Method to Check Onto Function

Find the range of $f(x)$ and show that range of $f(x) =$ codomain of $f(x)$.

- Any polynomial function of odd degree is always onto.
- The number of onto functions that can be defined from a finite set A containing n elements onto a finite set B containing 2 elements $= 2^n - 2$.
- If $n(A) \geq n(B)$, then number of onto function is 0.
- If A has m elements and B has n elements, where $m < n$, then number of onto functions from A to B is $n^m - {}^nC_1 (n-1)^m + {}^nC_2 (n-2)^m - \ldots, m < n$.
- f is said to be an **into function**, if there exists atleast one element in B having no pre-image in A. i.e. $f : A \rightarrow B$ is an into function, if it is not an onto function.



- f is said to be a **bijective function**, if it is one-one as well as onto.

NOTE
- If $f : A \rightarrow B$ is a bijective, then A and B have the same number of elements.
- If $n(A) = n(B) = m$, then number of bijective map from A to B is $m!$.

Composition of Functions

Let $f : A \rightarrow B$ and $g : B \rightarrow C$ are two functions. Then, the composition of f and g, denoted by

$gof : A \rightarrow C$, is defined as,

$gof\,(x) = g[f(x)], \forall\, x \in A$.

NOTE
- gof is defined only if $f(x)$ is an element of domain of g.
- Generally, $gof \neq fog$.

DAY PRACTICE SESSION 1

FOUNDATION QUESTIONS EXERCISE

1 If $Q = \left\{x : x = \frac{1}{y}, \text{ where } y \in N\right\}$, then

(a) $0 \in Q$ (b) $1 \in Q$ (c) $2 \in Q$ (d) $\frac{2}{3} \in Q$

2 If $P(A)$ denotes the power set of A and A is the void set, then what is number of elements in $P\{P\{P\{P(A)\}\}\}$?

(a) 0 (b) 1 (c) 4 (d) 16

3 If $X = \{4^n - 3n - 1 : n \in N\}$ and $Y = \{9(n-1) : n \in N\}$; where N is the set of natural numbers,then $X \cup Y$ is equal to

➜ **JEE Mains 2014**

(a) N (b) Y-X (c) X (d) Y

4 If A, B and C are three sets such that $A \cap B = A \cap C$ and $A \cup B = A \cup C$, then

(a) $A = C$ (b) $B = C$ (c) $A \cap B = \phi$ (d) $A = B$

5 Suppose $A_1, A_2, \ldots, A_{30}$ are thirty sets each having 5 elements and $B_1, B_2, \ldots, B_n$ are n sets each having 3 elements. Let $\bigcup_{i=1}^{30} A_i = \bigcup_{j=1}^{n} B_j = S$ and each element of S belongs to exactly 10 of A_i's and exactly 9 of B_j' s. The value of n is equal to

➜ **NCERT Exemplar**

(a) 15 (b) 3
(c) 45 (d) None of these

6 If A and B are two sets and $A \cup B \cup C = U$. Then, $\{(A-B) \cup (B-C) \cup (C-A)\}'$ is equal to

(a) $A \cup B \cup C$ (b) $A \cup (B \cap C)$
(c) $A \cap B \cap C$ (d) $A \cap (B \cup C)$

7 Let X be the universal set for sets A and B, if $n(A) = 200$, $n(B) = 300$ and $n(A \cap B) = 100$, then $n(A' \cap B')$ is equal to 300 provided $n(X)$ is equal to

(a) 600 (b) 700 (c) 800 (d) 900

8 If $n(A) = 1000$, $n(B) = 500$, $n(A \cap B) \geq 1$ and $n(A \cup B) = P$, then

(a) $500 \leq P \leq 1000$ (b) $1001 \leq P \leq 1498$
(c) $1000 \leq P \leq 1498$ (d) $1000 \leq P \leq 1499$

9 If $n(A) = 4$, $n(B) = 3$, $n(A \times B \times C) = 24$, then $n(C)$ is equal to

(a) 2 (b) 288 (c) 12 (d) 1

10 If $R = \{(3,3), (6,6), (9,9), (12,12), (6,12), (3,9), (3,12), (3,6)\}$ is a relation on the set $A = \{3, 6, 9, 12\}$. The relation is

(a) an equivalence relation
(b) reflexive and symmetric
(c) reflexive and transitive
(d) only reflexive

11 Let $R = \{(x, y) : x, y \in N \text{ and } x^2 - 4xy + 3y^2 = 0\}$, where N is the set of all natural numbers. Then, the relation R is

➜ **JEE Mains 2013**

(a) reflexive but neither symmetric nor transitive
(b) symmetric and transitive
(c) reflexive and symmetric
(d) reflexive and transitive

12 If $g(x) = 1 + \sqrt{x}$ and $f\{g(x)\} = 3 + 2\sqrt{x} + x$, then $f(x)$ is equal to

(a) $1 + 2x^2$ (b) $2 + x^2$
(c) $1 + x$ (d) $2 + x$

13 Let $f(x) = ax + b$ and $g(x) = cx + d$, $a \neq 0$, $c \neq 0$. Assume $a = 1$, $b = 2$, if $(fog)(x) = (gof)(x)$ for all x. What can you say about c and d?

(a) c and d both arbitrary (b) $c = 1$ and d is arbitrary
(c) c is arbitrary and $d = 1$ (d) $c = 1, d = 1$

14 If R is relation from $\{11, 12, 13\}$ to $\{8, 10, 12\}$ defined by $y = x - 3$. Then, R^{-1} is

(a) $\{(8, 11), (10, 13)\}$ (b) $\{(11, 18), (13, 10)\}$
(c) $\{(10, 13), (8, 11)\}$ (d) None of these

15 Let R be a relation defined by $R = \{(4, 5), (1, 4), (4, 6), (7, 6), (3, 7)\}$, then $R^{-1} OR$ is

(a) $\{(1, 1), (4, 4), (4, 7), (7, 4), (7, 7), (3, 3)\}$
(b) $\{(1, 1), (4, 4), (7, 7), (3, 3)\}$
(c) $\{(1, 5), (1, 6), (3, 6)\}$
(d) None of the above

16 Let A be a non-empty set of real numbers and $f : A \to A$ be such that $f(f(x)) = x$, $\forall\, x \in R$. Then, $f(x)$ is

(a) a bijection (b) one-one but not onto
(c) onto but not one-one (d) neither one-one nor onto

17 The function f satisfies the functional equation $3f(x) + 2f\left(\frac{x+59}{x-1}\right) = 10x + 30$ for all real $x \neq 1$. The value of $f(7)$ is

(a) 8 (b) 4 (c) – 8 (d) 11

18 The number of onto mapping from the set $A = \{1, 2, \ldots 100\}$ to set $B = \{1, 2\}$ is

(a) $2^{100} - 2$ (b) 2^{100} (c) $2^{99} - 2$ (d) 2^{99}

19 Let $f : R - \{n\} \to R$ be a function defined by $f(x) = \frac{x-m}{x-n}$, where $m \neq n$. Then,

(a) f is one-one onto (b) f is one-one into
(c) f is many-one onto (d) f is many-one into

20 A function f from the set of natural numbers to integers defined by $f(n) = \begin{cases} \frac{n-1}{2}, & \text{when } n \text{ is odd} \\ -\frac{n}{2}, & \text{when } n \text{ is even} \end{cases}$ is

(a) one-one but not onto (b) onto but not one-one
(c) both one-one and onto (d) neither one-one nor onto

21 Let $f: N \to N$ defined by $f(x) = x^2 + x + 1$, $x \in N$, then f is

(a) one-one onto (b) many-one onto
(c) one-one but not onto (d) None of these

22 Let R be the real line. Consider the following subsets of the plane $R \times R$.

$$S = \{(x, y): y = x + 1 \text{ and } 0 < x < 2\}$$

and $T = \{(x, y): x - y \text{ is an integer}\}$

Which one of the following is true?

(a) T is an equivalence relation on R but S is not
(b) Neither S nor T is an equivalence relation on R
(c) Both S and T are equivalence relations on R
(d) S is an equivalence relation on R but T is not

23 Consider the following relations

$R = \{(x, y) \mid x$ and y are real numbers and $x = wy$ for some rational number $w\}$;

$S = \left\{\left(\frac{m}{n}, \frac{p}{q}\right) \middle| m, n, p \text{ and } q \text{ are integers such that } n, q \neq 0 \text{ and } qm = pn\right\}$. Then,

(a) R is an equivalence relation but S is not an equivalence relation
(b) neither R nor S is an equivalence relation
(c) S is an equivalence relation but R is not an equivalence relation
(d) R and S both are equivalence relations

24 If $f(x) = \begin{cases} 2 + x, & x \geq 0 \\ 4 - x, & x < 0 \end{cases}$, then $f(f(x))$ is given by

(a) $f(f(x)) = \begin{cases} 4 + x, & x \geq 0 \\ 6 - x, & x < 0 \end{cases}$ (b) $f(f(x)) = \begin{cases} 4 + x, & x \geq 0 \\ x, & x < 0 \end{cases}$

(c) $f(f(x)) = \begin{cases} 4 - x, & x \geq 0 \\ x, & x < 0 \end{cases}$ (d) $f(f(x)) = \begin{cases} 4 - 2x, & x \geq 0 \\ 4 + 2x, & x < 0 \end{cases}$

25 **Statement I** A relation is defined by

$f(x) = \begin{cases} x^2, & 0 \leq x \leq 3 \\ 2x, & 3 \leq x \leq 9 \end{cases}$ is a function.

Statement II In a function, every member must have a unique image.

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true; Statement II is false
(d) Statement I is false; Statement II is true

DAY PRACTICE SESSION 2

PROGRESSIVE QUESTIONS EXERCISE

1 If $f(x) + 2f\left(\frac{1}{x}\right) = 3x$, $x \neq 0$ and $S = \{x \in R : f(x) = f(-x)\}$; then S → **JEE Mains 2016**

(a) is an empty set
(b) contains exactly one element
(c) contains exactly two elements
(d) contains more than two elements

2 $\left\{x \in R : \frac{2x - 1}{x^3 + 4x^2 + 3x} \in R\right\}$ is equal to

(a) $R - \{0\}$ (b) $R - \{0, 1, 3\}$
(c) $R - \{0, -1, -3\}$ (d) $R - \left\{0, -1, -3, \frac{1}{2}\right\}$

3 Given the relation $R = \{(1, 2)(2, 3)\}$ on the set $A = \{1, 2, 3\}$, the minimum number of ordered pairs which when added to R make it an equivalence relation is

(a) 5 (b) 7 (c) 6 (d) 8

4 The set $(A \cup B \cup C) \cap (A \cap B' \cap C')' \cap C'$ is equal to → **NCERT Exemplar**

(a) $B \cap C'$ (b) $A \cap C$
(c) $B' \cap C'$ (d) None of these

5 Let $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6\}$. Then the number of sets C such that $A \cap B \subseteq C \subseteq A \cup B$ is

(a) 6 (b) 9 (c) 8 (d) 10

6 On the set N of all natural numbers define the relation R by aRb iff the g.c.d. of a and b is 2, then R is

(a) reflexive but not symmetric (b) symmetric only
(c) reflexive and transitive (d) equivalence relation

7 Suppose f is a function satisfying $f(x + f(x)) = 4f(x)$ and $f(1) = 4$. The value of $f(21)$ is

(a) 16 (b) 64 (c) 4 (d) 44

8 Let $f\left(x + \frac{1}{x}\right) = x^2 + \frac{1}{x^2}$, $x \neq 0$, then $f(x)$ is equal to

(a) x^2 (b) $x^2 - 1$
(c) $x^2 - 2$ (d) $x^2 + 1$

9 Let $f(x) = \frac{x}{\sqrt{1 + x^2}}$, the $\underbrace{fofofo \ldots of}_{x \text{ times}}(x)$ is

(a) $\dfrac{x}{\sqrt{1 + \left(\sum_{r=1}^{n} r\right) x^2}}$ (b) $\dfrac{x}{\sqrt{1 + \left(\sum_{r=1}^{n} 1\right) x^2}}$

(c) $\left(\dfrac{x}{\sqrt{1 + x^2}}\right)^x$ (d) $\dfrac{nx}{\sqrt{1 + nx^2}}$

10 If two sets A and B are having 99 elements in common, then the number of elements common to each of the sets $A \times B$ and $B \times A$ are

(a) 2^{99} (b) 99^2 (c) 100 (d) 18

ANSWERS

SESSION 1	**1.** (b)	**2.** (d)	**3.** (d)	**4.** (b)	**5.** (c)	**6.** (c)	**7.** (b)	**8.** (d)	**9.** (a)	**10.** (c)
	11. (a)	**12.** (b)	**13.** (b)	**14.** (a)	**15.** (a)	**16.** (a)	**17.** (b)	**18.** (a)	**19.** (b)	**20.** (c)
	21. (c)	**22.** (a)	**23.** (c)	**24.** (a)	**25.** (d)					
SESSION 2	**1.** (c)	**2.** (c)	**3.** (b)	**4.** (a)	**5.** (c)	**6.** (b)	**7.** (b)	**8.** (c)	**9.** (b)	**10.** (b)

Hints and Explanations

SESSION 1

1 Clearly, $\frac{1}{y} \neq 0, 2$ and $\frac{2}{3}$ $[\because y \in N]$

$\therefore$ $\frac{1}{y}$ can be 1.

$\Rightarrow$ $x = 1 \in Q$

2 The number of elements in power set of A is 1.

$\therefore$ $P\{P(A)\} = 2^1 = 2$

$\Rightarrow$ $P\{P\{P(A)\}\} = 2^2 = 4$

$\Rightarrow$ $P\{P\{P\{P(A)\}\}\} = 2^4 = 16$

3 We have,

$X = \{4^n - 3n - 1 : n \in N\}$

$X = \{0, 9, 54, 243, \ldots\}$ [put $n = 1, 2, 3, \ldots$]

$Y = \{9(n-1) : n \in N\}$

$Y = \{0, 9, 18, 27, \ldots\}$ [put $n = 1, 2, 3, \ldots$]

It is clear that $X \subset Y$.

$\therefore$ $X \cup Y = Y$

4 Clearly, $A \cap B = A \cap C$ and $A \cup B = A \cup C$ possible if

$B = C$

5 Number of elements in $A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_{30}$ is 30×5 but each element is used 10 times, so

$n(S) = \frac{30 \times 5}{10} = 15$ ...(i)

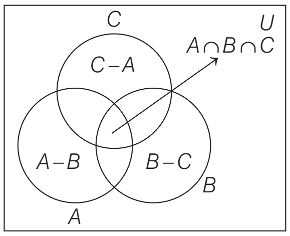
Similarly, number of elements in $B_1 \cup B_2 \ldots \cup B_n$ is $3\,n$ but each element is repeated 9 times, so

$n(S) = \frac{3n}{9}$

$\Rightarrow$ $15 = \frac{3n}{9}$ [from Eq. (i)]

$\Rightarrow$ $n = 45$

6 From Venn Euler's diagram,



It is clear that,

$\{(A - B) \cup (B - C) \cup (C - A)\}'$
$= A \cap B \cap C$

7 $\because n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$\therefore$ $n(A \cup B) = 200 + 300 - 100 = 400$

$\therefore$ $n(A' \cap B') = n(A \cup B)' = n(X) - n(A \cup B)$

$\Rightarrow$ $300 = n(X) - 400$

$\Rightarrow$ $n(X) = 700$

8 We know,

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$\therefore$ $P = 1500 - n(A \cap B)$

$\Rightarrow$ $n(A \cap B) = 1500 - P$

Clearly, $1 \leq n(A \cap B) \leq 500$

[$\because$ maximum number of elements common in A and $B = 500$]

$\Rightarrow$ $1 \leq 1500 - P \leq 500$

$\Rightarrow$ $-1499 \leq -P \leq -1000$

$\Rightarrow$ $1000 \leq P \leq 1499$

9 We know,

$n(A \times B \times C) = n(A) \times n(B) \times n(C)$

$\therefore$ $n(C) = \frac{24}{4 \times 3} = 2$

10 Since for each $a \in A$, $(a, a) \in R$. R is reflexive relation.

Now, $(6, 12) \in R$ but $(12, 6) \notin R$. So, it is not a symmetric relation.

Also, $(3, 6), (6, 12) \in R \Rightarrow (3, 12) \in R$

$\Rightarrow R$ is transitive.

11 $\because a^2 - 4a \cdot a + 3a^2 = 4a^2 - 4a^2 = 0$

$\therefore$ $(a, a) \in R, \forall a \in N \Rightarrow R$ is reflexive.

Now, as $a^2 - 4ab + 3b^2 = 0$

but $b^2 - 4ba + 3a^2 \neq 0$

$\therefore$ R is not symmetric.

Also, $(a, b) \in R$ and $(b, c) \in R$

$\not\Rightarrow$ $(a, c) \in N$

So, R is not transitive.

12 Given, $g(x) = 1 + \sqrt{x}$

and $f\{g(x)\} = 3 + 2\sqrt{x} + x$ ...(i)

$\Rightarrow f(1 + \sqrt{x}) = 3 + 2\sqrt{x} + x$

Put $1 + \sqrt{x} = y \Rightarrow x = (y - 1)^2$

$\therefore$ $f(y) = 3 + 2(y - 1) + (y - 1)^2$

$= 2 + y^2$

$\therefore$ $f(x) = 2 + x^2$

13 Here, $(fog)(x) = f\{g(x)\} = a(cx + d) + b$

and $(gof)(x) = g\{f(x)\} = c(ax + b) + d$

Since, $cx + d + 2 = cx + 2c + d$

$[\because a = 1, b = 2]$

Hence, $c = 1$ and d is arbitrary.

14 R is a relation from $\{11, 12, 13\}$ to $\{8, 10, 12\}$ defined by

$y = x - 3 \Rightarrow x - y = 3$

$\therefore$ $R = \{(11, 8), (13, 10)\}$

Hence, $R^{-1} = \{(8, 11), (10, 13)\}$

15 Clearly, $R^{-1} = \{(5, 4), (4, 1), (6, 4), (6, 7), (7, 3)\}$

Now, as $(4, 5) \in R$ and $(5, 4) \in R^{-1}$,

therefore $(4, 4) \in R^{-1} OR$

Similarly, $(1, 4) \in R$ and $(4, 1) \in R^{-1}$

$\Rightarrow$ $(1, 1) \in R^{-1} OR$
$(4, 6) \in R$ and $(6, 7) \in R^{-1}$

$\Rightarrow$ $(4, 7) \in R^{-1} OR$
$(7, 6) \in R$ and $(6, 7) \in R^{-1}$

$\Rightarrow$ $(7, 7) \in R^{-1} OR$
$(7, 6) \in R$ and $(6, 4) \in R^{-1}$

$\Rightarrow$ $(7,4) \in R^{-1}OR$
and $(3,7) \in R$ and $(7,3) \in R^{-1}$
$\Rightarrow$ $(3,3) \in R^{-1}OR$
Hence, $R^{-1}OR = \{(1,1),(4,4),(4,7),(7,7),(7,4),(3,3)\}$

16 Let $x, y \in A$ such that $f(x) = f(y)$, then
$$f(f(x)) = f(f(y))$$
$$\Rightarrow \quad x = y$$
$\Rightarrow f$ is one-one.
Also, for any $a \in A$, we have
$$f(f(a)) = a$$
$\Rightarrow f(b) = a$, where $b = f(a) \in A$
Thus, for each $a \in A$ (codomain) there exists $b = f(a) \in A$ such that $f(b) = a$
$\therefore$ f is onto.
Hence f is a bijective function.

17 We have, $3f(x) + 2f\left(\frac{x+59}{x-1}\right)$
$$= 10x + 30 \quad \dots \text{(i)}$$
On replacing x by $\frac{x+59}{x-1}$, we get
$$3f\left(\frac{x+59}{x-1}\right) + 2f(x) = \frac{40x+560}{x-1} \quad \dots \text{(ii)}$$
On solving Eqs. (i) and (ii), we get
$$f(x) = \frac{6x^2 - 4x - 242}{x-1}$$
$$\therefore \quad f(7) = \frac{6 \times 49 - 4 \times 7 - 242}{6} = 4$$

18 We know that if $n(A) = n$ and $n(B) = 2$, the number of onto relations from A to $B = 2^n - 2$
$\therefore$ Required number of relations
$$= 2^{100} - 2$$

19 Suppose for any $x, y \in R$,
$$f(x) = f(y)$$
$$\Rightarrow \quad \frac{x-m}{x-n} = \frac{y-m}{y-n}$$
$$\Rightarrow \quad x = y$$
So, f is one-one.
Let $\alpha \in R$ be such that $f(x) = \alpha$
$$\therefore \quad \frac{x-m}{x-n} = \alpha \Rightarrow x = \frac{m - n\alpha}{1-\alpha}$$
Clearly, $x \notin R$ for $\alpha = 1$
So, f is not onto.

20 Let $x, y \in N$ and both be even.
Then, $f(x) = f(y) \Rightarrow -\frac{x}{2} = -\frac{y}{2}$
$$\Rightarrow \quad x = y$$
Again, $x, y \in N$ and both are odd.
Then, $f(x) = f(y) \Rightarrow x = y$
So, f is one-one
Since, each negative integer is an image of even natural number and positive integer is an image of odd natural number. So, f is onto.

21 Let $x, y \in N$ such that $f(x) = f(y)$
$$\Rightarrow \quad x^2 + x + 1 = y^2 + y + 1$$
$$\Rightarrow \quad (x^2 - y^2) = y - x$$
$$\Rightarrow \quad (x-y)(x+y+1) = 0$$
$$\Rightarrow \quad x = y \text{ or } x = -y - 1 \notin N$$
$$\Rightarrow \quad x = y$$
$\Rightarrow$ f is one-one.
But f is not onto, as $1 \in N$ does not have any pre-image.
$\therefore$ f is one-one but not onto.

22 Since, $(1,2) \in S$ but $(2,1) \notin S$
Thus S is not symmetric.
Hence, S is not an equivalence relation.
Given, $\quad T = \{(x, y) : (x - y) \in I\}$
Now, $x - x = 0 \in I$, it is reflexive relation.
Again, now $(x - y) \in I$
$\Rightarrow y - x \in I$, it is symmetric relation.
Let $\quad x - y = I_1$
and $\quad y - z = I_2$
Then, $\quad x - z = (x - y) + (y - z)$
$$= I_1 + I_2 \in I$$
So, T is also transitive. Hence, T is an equivalence relation.

23 Since, the relation R is defined as $R = \{(x, y) \mid x, y$ are real numbers and $x = wy$ for some rational number $w\}$.
(a) **Reflexive** xRx as $x = 1x$
Here, $w = 1 \in$ Rational number
So, the relation R is reflexive.
(b) **Symmetric** $xRy \not\Rightarrow yRx$ as $0R1$ but $1 \not R 0$
So, the relation R is not symmetric.
Thus, R is not equivalence relation.
Now, for the relation S, defined as,
$$S = \left\{\left(\frac{m}{n}, \frac{p}{q}\right) \middle| m, n, p \text{ and } q \in \text{integers}\right.$$
such that $n, q \neq 0$ and $qm = pn\}$
(a) **Reflexive** $\frac{m}{n} S \frac{m}{n} \Rightarrow mn = mn$ [true]
Hence, the relation S is reflexive.
(b) **Symmetric** $\frac{m}{n} S \frac{p}{q} \Rightarrow mq = np$
$$\Rightarrow np = mq \Rightarrow \frac{p}{q} S \frac{m}{n}$$
Hence, the relation S is symmetric.
(c) **Transitive** $\frac{m}{n} S \frac{p}{q}$ and $\frac{p}{q} S \frac{r}{s}$
$$\Rightarrow \quad mq = np \text{ and } ps = rq$$
$$\Rightarrow mq \cdot ps = np \cdot rq \Rightarrow ms = nr$$
$$\Rightarrow \quad \frac{m}{n} = \frac{r}{s}$$
$$\Rightarrow \quad \frac{m}{n} S \frac{r}{s}$$
So, the relation S is transitive.
Hence, the relation S is equivalence relation.

24 Clearly,
$$f(f(x)) = \begin{cases} 2 + f(x), & f(x) \geq 0 \\ 4 - f(x), & f(x) < 0 \end{cases}$$
$$= \begin{cases} 2 + (2 + x), & x \geq 0 \\ 2 + (4 - x), & x < 0 \end{cases}$$
$$= \begin{cases} 4 + x, & x \geq 0 \\ 6 - x, & x < 0 \end{cases}$$

25 **Statement I** $f(x) = \begin{cases} x^2, & 0 \leq x \leq 3 \\ 2x, & 3 \leq x \leq 9 \end{cases}$
Now, $\quad f(3) = 9$
Also, $\quad f(3) = 2 \times 3 = 6$
Here, we see that for one value of x, there are two different values of $f(x)$.
Hence, it is not a function but Statement II is true.

SESSION 2

1 We have, $f(x) + 2f\left(\frac{1}{x}\right) = 3x$,
$$x \neq 0 \quad \dots \text{(i)}$$
$$\therefore \quad f\left(\frac{1}{x}\right) + 2f(x) = \frac{3}{x} \quad \dots \text{(ii)}$$
$\left[\text{replacing } x \text{ by } \frac{1}{x}\right]$
On multiplying Eq. (ii) by 2 and then subtracting it from Eq. (i), we get
$$-3f(x) = 3x - \frac{6}{x}$$
$$\Rightarrow \quad f(x) = \frac{2}{x} - x$$
Now, consider $f(x) = f(-x)$
$$\Rightarrow \frac{2}{x} - x = -\frac{2}{x} + x \Rightarrow \frac{4}{x} = 2x$$
$$\Rightarrow \quad x^2 = 2 \Rightarrow x = \pm\sqrt{2}$$
Thus, x contains exactly two elements.

2 Clearly, $\frac{2x-1}{x^3 + 4x^2 + 3x} \in R$ only when
$x^3 + 4x^2 + 3x \neq 0$
Consider $x^3 + 4x^2 + 3x = 0$
$$\Rightarrow \quad x(x^2 + 4x + 3) = 0$$
$$\Rightarrow \quad x(x+1)(x+3) = 0$$
$$\Rightarrow \quad x = 0, -1, -3$$
$$\therefore \quad \left\{x \in R : \frac{2x-1}{x^3 + 4x^2 + 3x} \in R\right\}$$
$$= R - \{0, -1, -3\}$$

3. For R to be an equivalence relation, R must be reflexive, symmetric and transitive.

R will be reflexive if it contains (1, 1), (2, 2) and (3, 3)

R will be symmetric if it contains (2, 1) and (3, 2)

R will be transitive if it contains (1, 3) and (3, 1)

Hence, minimum number of ordered pairs = 7

4 $(A \cup B \cup C) \cap (A \cap B' \cap C')' \cap C'$

$= (A \cup B \cup C) \cap (A' \cup B \cup C) \cap C'$

$= (\phi \cup B \cup C) \cap C'$

$= (B \cup C) \cap C'$

$= (B \cap C') \cup \phi = B \cap C'$

5 Here, $A \cap B = \{2, 4\}$

and $A \cup B = \{1, 2, 3, 4, 6\}$

$\because \ A \cap B \subseteq C \subseteq A \cup B$

$\therefore$ C can be {2, 4}, {1, 2, 4}, {3, 2, 4}, {6, 2, 4}, {1, 6, 2, 4}, {6, 3, 2, 4}, {1, 3, 2, 4}, {1, 2, 3, 4, 6}

Thus, number of set C which satisfy the given condition is 8.

6 Clearly, g.c.d $(a, a) = a, \forall \ a \in N$

$\therefore$ R is not reflexive.

If g.c.d $(a, b) = 2$, then g.c.d (b, a) is also 2.

Thus, $aRb \Rightarrow bRa$

Hence, R is symmetric.

According to given option, R is symmetric only.

7 We have,

$f(x + f(x)) = 4 f(x)$ and $f(1) = 4$

On putting $x = 1$, we get

$$f(1 + f(1)) = 4f(1)$$

$$\Rightarrow f(1 + f(1)) = 16$$

$$\Rightarrow f(1 + 4) = 16$$

$$\Rightarrow f(5) = 16$$

On putting, $x = 5$, we get

$$f(5 + f(5)) = 4f(5)$$

$$\Rightarrow f(5 + 16) = 4 \times 16$$

$$\Rightarrow f(21) = 64$$

8 We have,

$$f\left(x + \frac{1}{x}\right) = x^2 + \frac{1}{x^2}$$

$$= \left(x + \frac{1}{x}\right)^2 - 2$$

$$\Rightarrow f(x) = x^2 - 2$$

9 We have, $f(x) = \dfrac{x}{\sqrt{1 + x^2}}$

$$\Rightarrow f(f(x)) = \frac{f(x)}{\sqrt{1 + (f(x))^2}}$$

$$= \frac{\frac{x}{\sqrt{1 + x^2}}}{\sqrt{1 + \frac{x^2}{1 + x^2}}}$$

$$= \frac{x}{\sqrt{1 + 2x^2}}$$

Similarly, $f(f(f(x))) = \dfrac{x}{\sqrt{1 + 3x^2}}$

$\vdots \qquad \vdots$

$$\underbrace{fofo \ldots of}_{n \text{ times}} f(x) = \frac{x}{\sqrt{1 + nx^2}}$$

$$= \frac{x}{\sqrt{1 + \left(\sum_{r=1}^{n} 1\right) x^2}}$$

10 We know,

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$$

$$\therefore (A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$$

Thus, number of elements common to $A \times B$ and $B \times A$

$= n((A \times B) \cap (B \times A))$

$= n((A \cap B) \times (B \cap A))$

$= n(A \cap B) \times n(B \cap A)$

$= 99 \times 99 = 99^2$